Lecture No. 6

Example 2-D Problem

- Potential flow problems: Seepage in a granular soil. Assume that the soil is isotropic and Darcy's law gives:

$$v_x = K\frac{\partial u}{\partial x} \text{ and } v_y = K\frac{\partial u}{\partial y}$$

where

K = permeability coefficient

u = head

v_x and v_y are velocities in the *x* and *y* directions

- Continuity dictates that

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

- Substituting for v_x and v_y we have

$$L(u) = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

- Let's first check what the essential and natural b.c.'s are

$$\langle L(u), w\rangle = \iint\limits_{\Omega} K\left(u_{,xx} + u_{,yy}\right) w dx dy$$

- Integrate by parts using Green's theorem

$$\iint\limits_{\Omega} f \cdot g_{,x} dx dy = \int\limits_{\Gamma} f \cdot g \cos(n, x) d\Gamma - \iint\limits_{\Omega} f_{,x} g dx dy$$

Thus we integrate both terms by parts to obtain

$$\iint\limits_{\Omega} K\left(u_{,xx} + u_{,yy}\right) w dx dy = \int\limits_{\Gamma} K\left[u_{,x}\, \alpha_{nx} + u_{,y} \alpha_{ny}\right] w d\Gamma$$

$$- \iint\limits_{\Omega} \left[u_{,x} w_{,x} + u_{,y} w_{,y}\right] d\Omega$$

We note that

$$u_{,x}\alpha_{nx} + u_{,y}\alpha_{ny} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial n} = \frac{\partial u}{\partial n}$$

where $\frac{\partial x}{\partial n} = \alpha_{nx}$, $\frac{\partial y}{\partial n} = \alpha_{ny}$ has n is the unite normal to the boundary.

We note that the boundary term specifically has the following meaning

$$v_x = Ku_{,x} \text{ and } v_y = Ku_{,y}$$

thus

$$K\left(u_{,x}\alpha_{nx} + u_{,y}\alpha_{ny}\right) = v_x\alpha_{nx} + v_y\alpha_{ny}$$

$$= v_n$$

$$= normal\ velocity\left(K\frac{\partial u}{\partial n}\right)$$

Thus

$$\iint_{\Omega} K\left(u_{,xx} + u_{,yy}\right)wdxdy = \int_{\Gamma} K\frac{\partial u}{\partial n}wd\Gamma - \iint_{\Omega} K\left[u_{,x}w_{,x} + u_{,y}w_{,y}\right]d\Omega$$

- Thus the b.c.'s are as follows:
  - Essential b.c.'s

    $u = \overline{u}$ on Γ_E (specify potential head) on Γ_B

  - Natural b.c.'s

    $K\frac{\partial u}{\partial n} = \overline{v}_n$ on Γ_N (specify normal velocity)

- Let's now set up the fundamental weak form:

$$\langle \mathcal{E}_I, w_j \rangle_\Omega + \langle \mathcal{E}_{B,N}, w_j \rangle_{\Gamma_N} = 0 \qquad j = 1, N$$

$$\Rightarrow$$

$$\iint_\Omega K\left[u_{,xx} + u_{,yy}\right] w_j \, dx dy + \int_{\Gamma_N} \left[\overline{v}_n - K\frac{\partial u}{\partial n}\right] w_j \, d\Gamma = 0 \qquad j = 1, N$$

- Let's now integrate by parts:

$$-\iint_\Omega K\left[u_{,x} w_{j,x} + u_{,y} w_{j,y}\right] d\Omega + \int_\Gamma K\frac{\partial u}{\partial n} w_j \, d\Gamma + \int_{\Gamma_N} \left[\overline{v}_n - K\frac{\partial u}{\partial n}\right] w_j \, d\Gamma = 0 \qquad j = 1, N$$

- Now we expand the boundary terms:

$$-\iint_{\Omega} K\left[u_{,x}w_{j,x} + u_{,y}w_{j,y}\right]d\Omega + \int_{\Gamma_N} K\frac{\partial u}{\partial n}w_j d\Gamma + \int_{\Gamma_E} K\frac{\partial u}{\partial n}w_j d\Gamma$$

$$+ \int_{\Gamma_N} \overline{v}_n\, w_j d\Gamma - \int_{\Gamma_N} K\frac{\partial u}{\partial n}w_j d\Gamma = 0$$

Furthermore we note that $w_j = 0$ on Γ_E since all $\phi_i (= w_j)$ must satisfy the essential b.c's

- Thus the symmetrical weak form is:

$$-\iint_{\Omega} K\left(u_{,x}w_{j,x} + u_{,y}w_{j,y}\right)d\Omega + \int_{\Gamma_N} \overline{v}_n w_j\, d\Gamma = 0 \qquad j = 1, N$$

  - The space requirements for both ϕ_i and w_i are the same, $W^{(1)} = C_o$
  - We only require the essential b.c.'s to be satisfied on Γ_E

Time Dependent Problems

1. Use FD's for the time dependence and Galerkin for the spatial dependence. Applying the Galerkin methods in space leads to sets of time dependent o.d.e.'s. We resolve the time dependence in these equations with FD's. The time differencing scheme selected can control the success or failure of the method.

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i(t)\phi_i(\underline{x})$$

   The coefficients on $\alpha_i(t)$ are now time dependent.

2. Direct use of the Galerkin Approach in space and time:

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(\underline{x}, t)$$

   The interpolating functions, ϕ_i, are now functions in time as well as space.

Example

Let's look at the wave equation:

$$\lambda \nabla^2 u = \frac{\partial^2 u}{\partial t^2} \text{ in } \Omega$$

$$\Rightarrow$$

$$\lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial^2 u}{\partial t^2}$$

i.c.'s are $\quad u(\underline{x}, 0) = u_o(\underline{x})$

$$\frac{\partial u}{\partial t}(\underline{x}, 0) = \dot{u}_o(\underline{x})$$

b.c.'s are $\quad u(\underline{x}, t) = \overline{u}(\underline{x}, t)$ on Γ_1 (essential)

$$\lambda \frac{\partial u}{\partial n}(\underline{x}, t) = \overline{g}(\underline{x}, t) \text{ on } \Gamma_2 \text{ (natural)}$$

Approach 1

- Use Galerkin in space, FD in time
- Develop the fundamental weak weighted residual form:

$$\iiint_{\Omega} \left(\lambda \nabla^2 u - \frac{\partial^2 u}{\partial t^2}\right) w_j \, d\Omega - \iint_{\Gamma_N} \left(\lambda \frac{\partial u}{\partial n} - \overline{g}\right) w_j \, d\Gamma = 0 \qquad j = 1, N$$

This generates a differentially time dependent set of simultaneous algebraic equations with the time dependence in the $\alpha_i(t)$'s

$$\underline{\boldsymbol{A}} \frac{\partial^2 \alpha}{\partial t^2} + \underline{\boldsymbol{B}\, \alpha} = \underline{C}$$

Now we use the FD method to resolve this differential time dependence leading to a system of entirely algebraic equations. Thus we discretize $\frac{\partial^2 \underline{\alpha}}{\partial t^2}$.

Approach 2

- FE for time and space:

$$\int_{o}^{\Delta t} \left\{ \left(\iiint_{\Omega} \left(\lambda \nabla^2 u - \frac{\partial^2 u}{\partial t^2}\right) w_j \, d\Omega - \iint_{\Gamma_N} \left(\lambda \frac{\partial u}{\partial n} - \overline{g}\right) w_j \, d\Gamma \right) \right\} dt = 0$$